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Variability of Measures of Weapons Effectiveness

Volume VI: Estimation of the Weapons Parameters and Their Variances in the Carleton Damage Function

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For the three-parameter Carleton damage function, the techniques of linear and non-linear regressions are used to estimate the three parameters as well as their variance - covariance matrix. The computational aspect of the estimation is discussed, including language, program and subroutine. For 13 weapon/target situations, numerical results are provided.

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Estimation of the Weapons Parameters and Their Variances in the Carleton Damage Function

PREFACE

This report describes work done during 1983-1984 by Dr B. D. Sivazlian, principal investigator, Dr J. F. Mahoney, investigator, and Mr A. Black, graduate assistant, all from the Department of Industrial and Systems Engineering, the University of Florida, Gainesville, Florida 32611, under Contract No. F08635-83-C-0202 with the Air Force Armament Laboratory (AFATL), Armament Division, Eglin Air Force Base, Florida 32542. The program manager was Mr Daniel A. McInnis (DLYW).

This work addresses itself to the problem of estimating weapon parameters and the associated uncertainties when using the three-parameter Carleton damage function. This function approximates the probability of kill due to fragmentation of an exploding weapon in the absence of blast effect and delivery error. The techniques of linear and non-linear regressions are used to compute the estimates and their variance-covariance matrix.

The author has benefited from helpful discussions with Mr. Jerry Bass, Mr Daniel McInnis, and Mr Charles Reynolds who have contributed to the report through their comments. Dr Richard L. Scheaffer, Chairman, Department of Statistics, The University of Florida, has read an initial version of the manuscript and has made several useful suggestions.

The Public Affairs Office has reviewed this report, and it is releasable to the National Technical Information Service (NTIS), where it will be available to the general public, including foreign nationals.

This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Milton D. Kingcaid

MILTON D. KINGCAID, Colonel, USAF
Chief, Analysis and Strategic Defense Division

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SECTION I

INTRODUCTION

In this report, the problem of estimating the three parameters of the Carleton damage function is discussed. The technique of regression analysis is used to fit the damage function to a probability of kill (P_k) matrix with the use of the Statistical Analysis System (SAS) language [3],[4],[5]. The principal output of the study are the estimates D_0, R_x, and R_y in the damage function as well as their variance - covariance matrix, computed for a set of weapon/target situations.

The converse problem is also addressed. For a target located at (x,y) the estimated P_k as given by the Carleton damage function is computed as well as a measure of the error in P_k.

The report is divided into 11 sections. Section II gives the objective of the study. Section III describes the P_k matrix while Section IV digresses on the three-parameter Carleton damage function. In Section V and VI the various methodologies for estimating the weapons parameters are discussed. These include the Roger's (Snow) ellipse method, the linear regression method, and the non-linear regression method. Sections VII and VIII are related to the computational phases of the study (SAS language, programs, subroutines, etc.). The numerical results are exhibited in Section IX. In Section X, the Taylor's series estimation method is used to estimate P_k and to compute the error on P_k. Finally Section XI is a concluding section with recommendations.

SECTION II

OBJECTIVE OF THE STUDY

The purpose of this study is to determine estimates of the parameters of the generalized three-parameter Carleton damage function as well as the error bounds on these estimates. The mathematical representation of this function is

$$P_k = D_0 \exp[-D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})]$$

It is assumed that this function can approximate the value of the probability of kill appearing as elements of a matrix obtained experimentally as a result of weapon fragmentation analysis and target vulnerability.

A non-linear regression scheme is used to provide a best fit for the Carleton damage function. The output of the study consists in the following:

1. The estimates of the three parameters D_0, R_x, and R_y.
2. The variance-covariance matrix of D_0, R_x, and R_y.

The estimates obtained agree closely with the estimates using Roger's Ellipse Program. Since the mathematical basis of Roger's program cannot be identified and since the program does not generate the statistical errors on the estimates, it was necessary to use a non-linear regression technique to arrive at these errors.

Using the Carleton damage function and its parameter estimates, it is possible to determine for any weapon/target system the probability of kill, P_k, at any location (x,y) given that the weapon bursts at (0,0). In addition, using the methodology developed in [6], the variance-covariance matrix of D_0, R_x, and R_y can be used to obtain the error on the P_k value.

Parameter estimations were performed on a total of 13 weapon/target data sets. Some of the data sets were obtained using the same weapon/target system, but using different weapon impact angles. Different impact angles produce different P_k matrices, thus resulting in a Carleton damage function with different parameters.

SECTION III

DESCRIPTION OF THE P_k MATRIX

The P_k matrix is an array of numbers such that each element of the matrix provides the probability of kill P_k for a given point target positioned at a given location relative to the point at which the weapon bursts. It is assumed that:

1. the weapon bursts at (0,0);
2. the direction of the x-axis is the direction of range (vertical direction);
3. the direction of the y-axis is the direction of deflection (horizontal direction).

The maximum size of the P_k matrix printout is 40 x 40 with a total of 1600 data entries. All the entries are non-negative with values less than or equal to one. Some of the entries are simply zero. The columns of the array are headed by the ordinate distance between the point of weapon burst and the horizontal position of the center of the target. These distances are measured in feet, and the difference in distances between successive columns is constant. Similarly, the rows of the array are headed by the abscissa distance between the point of weapon burst and the vertical position of the center of the target. Again, these distances are measured in feet, and the difference in distances between successive rows is constant.

As would be expected, target points which are close to the weapon burst center have a high probability of kill. As the target points recede from the center, the probability of kill decreases until it reaches zero. It is

conceivable that even at the point (0,0) the probability of kill could be less than one; this is usually a characteristic of the target/weapon system under consideration.

In general, the P_k matrix will combine the effect of weapon fragmentation and blast. For the study under consideration, only fragmentation effects are accounted for.

On an actual P_k matrix, only the upper right and the lower right entries appear. The upper left and the lower left entries do not appear as it is usually assumed that the effect of the weapon burst is symmetrical about the range or x-axis.

In general, if the weapon impacts at 90 degrees, the P_k matrix is symmetrical about the deflection or y-axis. If, on the other hand, the weapon impacts at an angle less than 90 degrees, considerable asymmetry may result between the upper right entries and the lower right entries of the P_K matrix. The full display of an equiprobability curve as appearing in the P_k matrix resembles the wing contour of half a butterfly.

SECTION IV

THE CARLETON DAMAGE FUNCTION

The Carleton damage function is a mathematical formula which gives the probability of kill P_k of a weapon bursting at the point, (0,0) upon a target located at (x,y). Both weapon and target are idealized as points, and the only prevailing effect is due to weapon fragmentation. The formula is:

$$P_k = D_o \exp[-D_o(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})]$$

In its more general form, the function has three parameters:

1. The parameter R_x which is the weapon radius in the direction of range or x-axis and is measured in feet.
2. The parameter R_y which is the weapon radius in the direction of deflection or y-axis and is measured in feet.
3. The parameter D_0 which represents the maximum probability of kill which occurs at (0,0).

Of course, D_0 must always be a positive quantity whose value is less than or equal to one. For most of the weapons under consideration, $R_y > R_x$, since the fragmentation effect in the direction of deflection is significantly more marked than in the direction of range.

In general, the parameters of the Carleton damage function will depend on:

1. The impact angle of the weapon (in the direction of range).
2. The weapon characteristics.
3. The target characteristics.
4. The weapon velocity.

When P_k is plotted against x and y, the shape of the generated surface is that of a bivariate Gaussian distribution. The equiprobability contour lines are concentric ellipses with, in general, the major axis along the y-axis and the minor axis along the x-axis.

The theoretical values of P_k obtained from the damage function are such that they are always positive for any (x,y) values. That is, it is assumed that P_k is always positive irrespective of the magnitudes of x and y. In practice, however, the coverage area of the weapon due to fragmentation is finite and outside of a certain contour about the origin, $P_k=0$.

The theoretical equiprobability contour lines, which are concentric ellipses, approximate the experimentally obtained contour lines which are, in general, butterfly shaped. Thus, in using the P_k obtained from the Carleton function at a given (x,y), a discrepancy will, in general, result between the true mean value of P_k (the observed sample mean) and the calculated mean value of P_k which is the theoretical $E[P_k]$ from the model. This difference is reflected in terms of a standard deviation σ_{P_k}. Thus, the expression

$$E[P_k] \pm 2\sigma_{P_k}$$

will state that in at least 75 percent of the cases (using Chebyshev's theorem) the true mean value of P_k will be within the above range. σ_{P_k} can be calculated (see Section X) explicitly as a mathematical formula involving the estimates of D_0, R_x, and R_y as well as their variance-covariance matrix. The numerical value of σ_{P_k} can then be obtained using the outputs of the non-linear regression analysis which provides the numerical values of D_0, R_x, and R_y as well as their variance-covariance matrix.

SECTION V

ESTIMATION OF THE WEAPONS PARAMETERS (1)

In this section Roger's ellipse program is briefly discussed and a summary of the linear regression procedure is presented. The non-linear regression procedure based on the Gauss-Newton method is discussed in detail in Section VI.

1. Roger's Ellipse Program

To estimate the parameters D_0, R_x and R_y of the Carleton damage function, Mr. Roger Snow, formerly with the RAND Corporation, developed an algorithm which appears in a program listing as a subroutine under the heading "Roger's Ellipse Program." Unfortunately, the subroutine is undocumented, and it was not possible to derive the methodology, the mathematical rationale, and the equations leading to the algorithm. The program subroutine provides a single estimate number for D_0, R_x, and R_y. Despite the fact that the Carleton damage function specifies a value of D_0 in the range $0 < D_0 \leq 1$, the algorithm has sometimes generated a value of $D_0 > 1$, which, in effect, means that the largest probability of kill obtained analytically may take on values greater than one. Finally, the ellipse program does not compute the errors associated with the estimation of the three parameters. (See Appendix A for a listing of the program.)

2. Linear Regression Method

Recall that the expression for the Carleton damage function is

$$P_k = D_0 \exp[-D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})]$$

By taking natural logarithms on both sides of this expression one obtains

$$\ln P_k = \ln D_0 - \frac{D_0}{R_x^2} x^2 - \frac{D_0}{R_y^2} y^2$$

This can be made a linear expression in $u=x^2$ and $v=y^2$ and, hence, $\ln D_0$, $(-D_0/R_x^2)$ and $(-D_0/R_y^2)$ can be estimated using the classical techniques of linear regression. Details of the methodology when $D_0=1$ is given in [6].

Unfortunately, the technique is ill-suited to provide accurate estimates of the parameters under consideration. There are three main factors which contribute to this situation:

a. Since P_k lies in the range $0 < P_k < 1$, $\ln P_k$ becomes very sensitive to small variations in P_k.

b. Data entries for which $P_k=0$ in the matrix must be discarded since they generate infinite values for $\ln P_k$.

c. Points which are far from the origin are weighted more strongly than are those points near the origin. Hence, a poor fit near the origin results.

To see this, note that the sum of squares of the residuals for the linearized model has the form

$$\sum_{i=1}^{n} (\ln P_{k_i} - \ln P_{k_i}^*)^2$$

where n = number of observations;

P_{k_i} = calculated value of the i^{th} data point;

$P^*_{k_i}$ = experimental measured value of the i^{th} data point.

Let

$$\delta P_{k_i} = P_{k_i} - P^*_{k_i}$$

then note that

$$\ln P_{k_i} - \ln P^*_{k_i} = \ln \left(\frac{P_{k_i}}{P^*_{k_i}}\right)$$

$$= \ln\left(\frac{P^*_{k_i} + \delta P_{k_i}}{P^*_{k_i}}\right)$$

$$= \ln \left(1 + \frac{\delta P_{k_i}}{P^*_{k_i}}\right)$$

If the δP_{k_i}'s are small, then

$$\ln \left(1 + \frac{\delta P_{k_i}}{P^*_{k_i}}\right) = \frac{\delta P_{k_i}}{P^*_{k_i}}$$

Hence, the sum of squares of the residuals is given approximately by

$$\sum_{i=1}^{n} \left(\frac{\delta P_{k_i}}{P^*_{k_i}}\right)^2$$

On the other hand, the sum of squares of the residuals for the non-linear regression model is simply

$$\sum_{i=1}^{n} (\delta P_{k_i})^2$$

The $(1/P^*_{k_i})^2$ factor in the linearized model acts as a weighting factor which gives additional weight to those points for which $P^*_{k_i}$ is small, i.e., far from the origin.

The main advantage of the method is that analytic techniques are available to obtain closed form expressions for the estimates, as well as their variance-covariance matrix. A more accurate technique is that of non-linear regression which will be discussed later.

The linear regression method, however, may be used to obtain an initial estimate of the parameters which then can be used as a starting point for the non-linear regression technique. To simplify the computational routine, an initial guess of D_0 may be taken as unity. This reduces the transformed equation to the form

$$Z = au + bv$$

where $Z = \ln P_k^{-1}$, $u = x^2$, $v = y^2$, $a = 1/R_x^2$, and $b = 1/R_y^2$. The techniques for estimating a and b are classical and are elaborated in [6].

We present a summary of the results. Let $\bar{a}$, $\bar{b}$, $\bar{R}_x$ and $\bar{R}_y$ be the estimates of a, b, R_x, and R_y, respectively. Then from [6].

$$\bar{R}_x = \frac{1}{\sqrt{\bar{a}}} + \frac{3}{8\,\bar{a}^{5/2}} \text{Var}[a]$$

$$\bar{R}_y = \frac{1}{\sqrt{\bar{b}}} + \frac{3}{8\,\bar{a}^{5/2}} \text{Var}[a]$$

$$\text{Var}[R_x] = \frac{1}{4\,\bar{a}^3} \text{Var}[a]$$

$$\text{Var}[R_y] = \frac{1}{4\,\bar{b}^3} \text{Var}[b]$$

$$\text{Cov}[R_x, R_y] = \frac{1}{4\,(\bar{a}\,\bar{b})^{3/2}} \text{Cov}[a,b]$$

SECTION VI

ESTIMATION OF THE WEAPONS PARAMETERS (2)

In this section, a non-linear regression procedure based on the Gauss-Newton technique is discussed (see e.g., [1],[2]).

1. Non-Linear Regression Method

Suppose that it is required to fit a function of the form

$$p_i \doteq M(x_i, y_i;\ A, B, C) \tag{1}$$

$i=1,2,\ldots,n$, to a large collection of n data. Here x_i and y_i are a pair of independent variables, and p_i is the corresponding dependent variable. A,B, and C are the parameters of the model whose values are to be estimated through a least square procedure.

The residual at each point i, $i=1,2,\ldots,n$, is defined by

$$r_i = M(x_i, y_i;\ A, B, C) - p_i \tag{2}$$

Then, the parameters A,B, and C are determined by minimizing the sum of squares of the residuals. For mathematical convenience, the objective function adopted has the form

$$f(A,B,C) = \frac{1}{2} \sum_{i=1}^{n} r_i^2 \tag{3}$$

To minimize this function, set its partial derivatives with respect to A,B, and C to zero and thus obtain

$$\sum_{i=1}^{n} r_i \frac{\partial r_i}{\partial A} = 0 \tag{4}$$

$$\sum_{i=1}^{n} r_i \frac{\partial r_i}{\partial B} = 0 \tag{5}$$

$$\sum_{i=1}^{n} r_i \frac{\partial r_i}{\partial C} = 0 \tag{6}$$

Using (2) and for brevity, setting M_i equal to $M(x_i, y_i, A, B, C)$, equations (4), (5), and (6) become

$$\sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial A} = 0 \tag{7}$$

$$\sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial B} = 0 \tag{8}$$

$$\sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial C} = 0 \tag{9}$$

These last three equations must now be solved for A,B, and C. Generally, these equations are non-linear and a standard method of solution is the iterative method of Newton.

For the moment define

$$F(A,B,C) = \sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial A} \tag{10}$$

$$G(A,B,C) = \sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial B} \tag{11}$$

$$H(A,B,C) = \sum_{i=1}^{n} (M_i - p_i) \frac{\partial M_i}{\partial C} \tag{12}$$

Let A^*, B^*, and C^* be the solution to the simultaneous set of equations

$$F(A,B,C) = 0 \tag{13}$$

$$G(A,B,C) = 0 \tag{14}$$

$$H(A,B,C) = 0 \tag{15}$$

Also, let A^o, B^o, and C^o be an approximate solution of (13) and (14) and (15), while A^+, B^+, and C^+ represent an approximate solution to (13), (14), and, (15) which is an improvement over A^o, B^o, C^o.

By Taylor's series

$$F(A^*, B^*, C^*) = 0 = F(A^o, B^o, C^o) + (A^* - A^o) \frac{\partial F}{\partial A}$$

$$+ (B^* - B^o) \frac{\partial F}{\partial B} + (C^* - C^o) \frac{\partial F}{\partial C} + \ldots \tag{16}$$

$$G(A^*, B^*, C^*) = 0 = G(A^o, B^o, C^o) + (A^* - A^o) \frac{\partial G}{\partial A}$$

$$+ (B^* - B^o) \frac{\partial G}{\partial B} + (C^* - C^o) \frac{\partial G}{\partial C} + \ldots \tag{17}$$

$$H(A^*, B^*, C^*) = 0 = H(A^o, B^o, C^o) + (A^* - A^o) \frac{\partial H}{\partial A}$$

$$+ (B^* - B^o) \frac{\partial H}{\partial B} + (C^* - C^o) \frac{\partial H}{\partial C} + \ldots \tag{18}$$

where all the partial derivatives are computed at the point (A^o, B^o, C^o). If the right sides of (16), (17), and (18) are retained as infinite series, one could solve in principle for A^*, B^*, and C^*. However, if the series are truncated just before the second derivative terms, the equations cannot be solved for A^*, B^*, and C^*, but instead may be solved for A^+, B^+, and C^+. This is easily done since the resulting equations are linear. Writing the truncated versions of (16), (17), and (18) in matrix form, one obtains at the point (A^o,B^o,C^o)

$$\begin{bmatrix} \frac{\partial F}{\partial A} & \frac{\partial F}{\partial B} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial A} & \frac{\partial G}{\partial B} & \frac{\partial G}{\partial C} \\ \frac{\partial H}{\partial A} & \frac{\partial H}{\partial B} & \frac{\partial H}{\partial C} \end{bmatrix} \begin{bmatrix} A^+ - A^o \\ B^+ - B^o \\ C^+ - C^o \end{bmatrix} = \begin{bmatrix} -F \\ -G \\ -H \end{bmatrix} \tag{19}$$

By means of (10), (11), and (12), the quantities F, G, and H may be removed; consequently, the matrix on the left-hand side of (19) may be written as the sum of two matrices, namely,

$$\begin{bmatrix} \frac{\partial F}{\partial A} & \frac{\partial F}{\partial B} & \frac{\partial F}{\partial C} \\ \frac{\partial G}{\partial A} & \frac{\partial G}{\partial B} & \frac{\partial G}{\partial C} \\ \frac{\partial H}{\partial A} & \frac{\partial H}{\partial B} & \frac{\partial H}{\partial C} \end{bmatrix} = \underline{S} + \underline{T} \tag{20}$$

where

$$\underline{S} = \sum_{i=1}^{n} (M_i - p_i) \begin{bmatrix} \frac{\partial^2 M_i}{\partial A^2} & \frac{\partial^2 M_i}{\partial A \partial B} & \frac{\partial^2 M_i}{\partial A \partial C} \\ \frac{\partial^2 M_i}{\partial B \partial A} & \frac{\partial^2 M_i}{\partial B^2} & \frac{\partial^2 M_i}{\partial B \partial C} \\ \frac{\partial^2 M_i}{\partial C \partial A} & \frac{\partial^2 M_i}{\partial C \partial B} & \frac{\partial^2 M_i}{\partial C^2} \end{bmatrix} \tag{21}$$

and

$$\underline{T} = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial M_i}{\partial A}\frac{\partial M_i}{\partial A} & \frac{\partial M_i}{\partial A}\frac{\partial M_i}{\partial B} & \frac{\partial M_i}{\partial A}\frac{\partial M_i}{\partial C} \\ \frac{\partial M_i}{\partial B}\frac{\partial M_i}{\partial A} & \frac{\partial M_i}{\partial B}\frac{\partial M_i}{\partial B} & \frac{\partial M_i}{\partial B}\frac{\partial M_i}{\partial C} \\ \frac{\partial M_i}{\partial C}\frac{\partial M_i}{\partial A} & \frac{\partial M_i}{\partial C}\frac{\partial M_i}{\partial B} & \frac{\partial M_i}{\partial C}\frac{\partial M_i}{\partial C} \end{bmatrix} \tag{22}$$

The right-hand side of expression (19) may be written as

$$\begin{bmatrix} F \\ G \\ H \end{bmatrix} = \sum_{i=1}^{n} (M_i - p_i) \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \frac{\partial M_i}{\partial B} \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \equiv \underline{b} \tag{23}$$

In terms of $\underline{S}$, $\underline{T}$ and $\underline{b}$, (19) becomes

$$(\underline{S} + \underline{T}) \begin{bmatrix} A^+ - A^o \\ B^+ - B^o \\ C^+ - C^o \end{bmatrix} = - \underline{b} \tag{24}$$

After solution one gains

$$\begin{bmatrix} A^+ \\ B^+ \\ C^+ \end{bmatrix} = \begin{bmatrix} A^o \\ B^o \\ C^o \end{bmatrix} - (\underline{S} + \underline{T})^{-1} \underline{b} \tag{25}$$

where it is understood that the right-hand side of (25) is evaluated at the point (A^o, B^o, C^o).

From (25), one obtains A^+, B^+, and C^+ which are now redefined as A^o, B^o, C^o, and the process is repeated until the vectors $[A^o, B^o, C^o]$ and $[A^+, B^+, C^+]$ are the same, or at least until they become arbitrarily close to each other. The amount of work is staggering, and simplifications are sought.

2. Case when M(x,y; A,B,C) is linear

If M(x,y; A,B,C) is linear in A,B, and C, then two important simplifications result.

a. $\underline{S}$ becomes zero, and

b. $M_i \equiv M_i\ (x_i, y_i;\ A^o, B^o, C^o)$

$$= A^o \frac{\partial M_i}{\partial A} + B^o \frac{\partial M_i}{\partial B} + C^o \frac{\partial M_i}{\partial C} \tag{26}$$

Setting $\underline{S}$ to zero and multiplying by $\underline{T}$ transforms (25) into:

$$\underline{T}\begin{bmatrix} A^+ \\ B^+ \\ C^+ \end{bmatrix} = \underline{T}\begin{bmatrix} A^o \\ B^o \\ C^o \end{bmatrix} - \sum_{i=1}^{n} M_i \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \frac{\partial M_i}{\partial B} \\ \frac{\partial M_i}{\partial C} \end{bmatrix}$$

$$+ \sum_{i=1}^{n} P_i \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \frac{\partial M_i}{\partial B} \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \qquad (27)$$

But from (22), it follows that

$$\underline{T} = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \frac{\partial M_i}{\partial B} \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \left[\frac{\partial M_i}{\partial A}, \quad \frac{\partial M_i}{\partial B}, \quad \frac{\partial M_i}{\partial C}\right] \qquad (28)$$

from which it follows that

$$\underline{T}\begin{bmatrix} A^o \\ B^o \\ C^o \end{bmatrix} = \sum_{i=1}^{n} \left[A^o \frac{\partial M_i}{\partial A} + B^o \frac{\partial M_i}{\partial B} + C^o \frac{\partial M_i}{\partial C}\right] \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \frac{\partial M_i}{\partial B} \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \qquad (29)$$

For the linear case, because of (26), this is the same as

$$\underline{I} \begin{bmatrix} A^0 \\ \\ B^0 \\ \\ C^0 \end{bmatrix} = \sum_{i=1}^{n} M_i \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \\ \frac{\partial M_i}{\partial B} \\ \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \qquad (30)$$

Hence, the first two terms on the right side of (27) cancel yielding

$$\underline{I} \begin{bmatrix} A^+ \\ \\ B^+ \\ \\ C^+ \end{bmatrix} = \sum_{i=1}^{n} p_i \begin{bmatrix} \frac{\partial M_i}{\partial A} \\ \\ \frac{\partial M_i}{\partial B} \\ \\ \frac{\partial M_i}{\partial C} \end{bmatrix} \qquad (31)$$

Thus, A^+, B^+, and C^+ may be solved for without iteration. These values are in fact A^*, B^* and C^*.

3. <u>The Gauss-Newton Method</u>

Returning to the case when $M(x,y;\ A,B,C)$ is not linear in A, B, and C, one may use (25), in the iterative fashion outlined, in order to approximate A^*, B^*, and C^*. This method is tedious and has been called the full-Newton method. As the full-Newton method proceeds, $\underline{b}$ becomes closer and closer to zero. From (21), one may see that $\underline{S}$ is linear in the residuals $(M_i - p_i)$. Since it is reasonable to suppose that these residuals decrease during the iteration process, it follows that $\underline{S}$ becomes less and less important as the iterations

evolve. The Gauss-Newton method exploits this notion by setting $\underline{S}$ equal to zero. Hence, the Gauss-Newton method uses the formula

$$\begin{bmatrix} A^+ \\ B^+ \\ C^+ \end{bmatrix} = \begin{bmatrix} A^0 \\ B^0 \\ C^0 \end{bmatrix} - \underline{T}^{-1}\,\underline{b} \tag{32}$$

in an iterative fashion. That is, A^0, B^0 and C^0 are used to evaluate the right hand side of (32) and, hence, to calculate A^+, B^+ and C^+. Next, A^0, B^0 and C^0 are replaced by the A^+, B^+ and C^+ that were just computed. The process continues until the vectors $[A^+,B^+,C^+]$ and $[A^0,B^0,C^0]$ become arbitrarily close to each other. The Gauss-Newton method does not involve the use of second derivatives.

4. Remark

Before passing on to an example, let it be noted that some authorities define the matrix

$$\underline{J} = \begin{bmatrix} \frac{\partial M_1}{\partial A} & \frac{\partial M_1}{\partial B} & \frac{\partial M_1}{\partial C} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial M_n}{\partial A} & \frac{\partial M_n}{\partial B} & \frac{\partial M_n}{\partial C} \end{bmatrix} \tag{33}$$

From (22), $\underline{T}$ is easily computed as

$$\underline{T} = \underline{J}^T\,\underline{J} \tag{34}$$

Also, if one defines the vectors

$$\underline{p} = \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \tag{35}$$

and

$$\underline{M} = \begin{bmatrix} M_1 \\ \cdot \\ \cdot \\ \cdot \\ M_n \end{bmatrix} \tag{36}$$

one immediately obtains from (23)

$$\underline{b} = \underline{J}^T [\underline{M} - \underline{p}] \tag{37}$$

The vector $[\underline{M} - \underline{p}]$ is called the residual vector. The root mean square (RMS) associated with the residual vector is defined by

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{n} (M_i - p_i)^2}{n}} = \sqrt{\frac{2f}{n}}$$

where f is the value of the objective function given in (3) and n is the number of observation in the data set. The RMS is an aggregate measure of the residuals, and as the iterations evolve, the value of the RMS should decrease and converge to a fixed value.

5. Example

Let the function in question which is to be used to fit the data be

$$P_k = D_0 \exp[-D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})] \tag{38}$$

Let the given data be

i	x_i	y_i	P_{k_i}
1	0	0	.9
2	0	1	.5
3	1	0	.6
4	1	1	.3

The parameters to be estimated are D_0, R_x and R_y. To obtain the initial values of the parameters, namely D_0^o, R_x^o and R_y^o, use a linear model. To this end, take the natural logarithm of (38) and obtain:

$$\ln P_k = \ln D_0 - \frac{D_0}{R_x^2} x^2 - \frac{D_0}{R_y^2} y^2 \tag{39}$$

An initial value of D_0 may be taken as $D_0=1$. However, this is not necessary as will be noted in the sequel. Let

$$\alpha = \ln D_0, \quad a = -\frac{D_0}{R_x^2}, \quad b = -\frac{D_0}{R_y^2} \tag{40}$$

Thus

$$D_0 = e^{\alpha} \tag{41}$$

$$R_x = \sqrt{-\frac{D_0}{a}} \tag{42}$$

$$R_y = \sqrt{-\frac{D_0}{b}} \tag{43}$$

For the linear problem

$$\begin{aligned} \ln P_k &= \alpha + ax^2 + by^2 \\ &= M(x^2, y^2; \alpha, a, b) \end{aligned} \tag{43}$$

where x^2 and y^2 are the independent variables and $\ln P_k$ is the dependent variable. Hence,

$$\begin{bmatrix} 1 & x_1^2 & y_1^2 \\ 1 & x_2^2 & y_2^2 \\ 1 & x_3^2 & y_3^2 \\ 1 & x_4^2 & y_4^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

From (34)

$$\underline{T} = \underline{J}^T J = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

From (35)

$$\underline{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \ln P_{k_1} \\ \ln P_{k_2} \\ \ln P_{k_3} \\ \ln P_{k_4} \end{bmatrix} = \begin{bmatrix} -.1053605 \\ -.6931472 \\ -.5108256 \\ -1.2039728 \end{bmatrix}$$

The right-hand side of (31) is

$$\underline{J}^T\underline{p} = \begin{bmatrix} -2.513306124 \\ -1.714798428 \\ -1.897119985 \end{bmatrix}$$

Hence, (31) becomes

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha^+ \\ a^+ \\ b^+ \end{bmatrix} = \begin{bmatrix} -2.513306124 \\ -1.714798428 \\ -1.897119985 \end{bmatrix}$$

The solution is

$$\alpha^+ = -.0790203868$$

$$a^+ = -.4581453659$$

$$b^+ = -.6404669227$$

From (41), (42), and (43) it follows that

$$D_0^o = .9240210864$$

$$R_x^o = 1.420166592 \tag{45}$$

$$R_y^o = 1.201137119$$

We first compute the RMS value for the non-linear function (38). The $\underline{M}$ vector is obtained by substituting (45) in (38) and calculating it for each i=1,2,3,4 to yield

$$\underline{M} = \begin{bmatrix} .9240210864 \\ .4870018732 \\ .5844022479 \\ .3080070289 \end{bmatrix}$$

The $\underline{p}$ vector is immediately given from the data set. The logarithmic transformation is not used for the non-linear function.

$$\underline{p} = \begin{bmatrix} .9 \\ .5 \\ .6 \\ .3 \end{bmatrix}$$

The residual vector for (D_0^o, R_x^o, R_y^o) is

$$\underline{M} - \underline{p} = \begin{bmatrix} .0240210864 \\ -.0129981268 \\ -.0155977521 \\ .0080070289 \end{bmatrix}$$

This results in a RMS value of .0162278023. Values in (45) will now be used as initial values in the Gauss-Newton method where the non-linear function (38) is being employed. Note that

$$\frac{\partial M}{\partial D_0} = \frac{\partial P_k}{\partial D_0} = [1 - D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})] \exp[- D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^x})] \qquad (46)$$

$$\frac{\partial M}{\partial R_x} = \frac{\partial P_k}{\partial R_x} = 2x^2 \ \frac{D_0^2}{R_x^3} \exp[- D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})] \qquad (47)$$

$$\frac{\partial M}{\partial R_y} = \frac{\partial P_k}{\partial R_y} = 2y^2 \ \frac{D_0^2}{R_y^3} \exp[- D_0(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})] \qquad (48)$$

The new $\underline{J}$ matrix as defined in (33) is now formed using (46), (47), and (48), all evaluated at the point (D_0^o, R_x^o, R_y^o) as given by (45).

$$\underline{J} = \begin{bmatrix} 1 & 0 & 0 \\ .1894905697 & 0 & .5193555108 \\ .3426989609 & .3770560205 & 0 \\ -.0328707630 & 1987259717 & .3284692657 \end{bmatrix}$$

Thus the new $\underline{T}$ is

$$\underline{T} = \underline{J}^T\underline{J} = \begin{bmatrix} 1.154429741 & .1226844321 & .087659362 \\ .1226844321 & .1816632544 & .065275374 \\ .0876159362 & .065275374 & .3776222051 \end{bmatrix} \tag{49}$$

From (37) one obtains

$$\underline{b} = \underline{J}^T [\underline{M} - \underline{\rho}] = \begin{bmatrix} .0159495334 \\ -.0042900217 \\ -.0041205859 \end{bmatrix} \tag{50}$$

Using (32), the equation for an improved value of the parameters may be written as:

$$\begin{bmatrix} D_0^+ \\ R_x^+ \\ R_y^+ \end{bmatrix} = \begin{bmatrix} D_0^o \\ R_x^o \\ R_y^o \end{bmatrix} - \underline{T}^{-1}\,\underline{b} \tag{51}$$

Substituting (45), (49), and (50) in (51) results in

$$\begin{aligned} D_0^+ &= .9060476147 \\ R_x^+ &= 1.452509594 \\ R_y^+ &= 1.210628475 \end{aligned} \tag{52}$$

Using (52) a new $\underline{M}$ and $\underline{p}$ vector can be calculated resulting in a residual vector of

$$\underline{M} - \underline{p} = \begin{bmatrix} .0060476147 \\ -.0117187842 \\ -.0102837618 \\ .0178059929 \end{bmatrix}$$

The corresponding RMS is .0122138516.

Thus, one iteration of the Gauss-Newton method has been completed. As can be seen, the RMS value associated with the residual vector decreased from .0162278023 to .0122138516 which indicates that the method is converging.

6. The Variance - Covariance Matrix

Whereas, in the linear least-square model the variance-covariance matrix of the parameter estimates was uniquely defined and could be derived analytically, the situation is quite different for non-linear least squares. In non-linear least-squares several corresponding variance-covariance matrices are used, all of them being approximations. In addition, they are not as reliable as the linear least squares one. In fact, some authors (see e.g., [1]) even caution users of non-linear least-squares software about an indiscriminate use of the variance-covariance estimates.

One particular variance-covariance matrix that is often used is $(RMS)^2\ \underline{I}_f$, where $\underline{I}_f$ is the final value of the matrix (34) computed at the final estimate values of the parameters, and (RMS) is the root mean square associated with the residuals.

SECTION VII

COMPUTATIONAL ASPECTS OF THE PROBLEM

1. Program Development

During the development of the programs, factors that were considered were:

(a) Accuracy of results

Would significant error result due to roundoff, particularly due to the large number of complex mathematical operations required on a large data set? Conceivably, a small roundoff error occurring at the beginning could propagate and result in major errors in the estimates of the parameters.

(b) Simplicity and flexibility

The procedures developed should be easy to understand and be adaptable to several computer facilities.

(c) Data management and additional analysis features

Adequacy of data storage should be a prime requirement. Also, the programmer/analyst might need to perform additional analyses, plots, charts, etc.

With these considerations in mind, it was decided to use the Statistical Analysis System (SAS) language to perform the desired linear and non-linear regression analyses. For detailed description the reader is referred to [5].

For the linear regression case, the PROC REG procedure was used. This procedure is based on the least squares principle to produce Best Linear Unbiased Estimates (BLUE) under classical statistical assumptions. For the non-linear regression case, the PROC NLIN procedure was used. Several non-linear estimation methods are available using this procedure (e.g., Gauss-Newton, Marquardt and steepest descent).

2. Program Performance

Two sets of programs, one for the linear case and the other for the nonlinear case, were developed. To check the degree of roundoff error, a 15-observation data set was analyzed using the PROC REG procedure. The parameter estimates and computed variances/covariances were in agreement with hand-calculated results to the eighth decimal place. The programs were then tested on a full size (1,600-observation) data set and produced output in the desired format.

Two very short subroutines were written and used to plot the residuals (observed P_k - predicted P_k) versus the x(range) values or the y (deflection) values and to provide a printout of the residuals. The plotting routine was used extensively in evaluating output result.

For the PROC NLIN procedure, both the Gauss-Newton and the Marquardt methods were used on a full-size data set with results in agreement to the third decimal place for the parameter estimates and the eighth decimal place for the variance - covariance matrix. The Gauss-Newton method, as described in Section VI, was the one that was arbitrarily selected and used on the rest of the data set.

Finally, a number of runs under the PROC NLIN procedure, using arbitrary extreme starting values for the parameters (PARMS statement), indicated that close starting values for the R_x and R_y parameters were not necessary for accurate results, but might lead to significant savings in computer time. However, different starting values for the D_0 parameter sometimes resulted in a non-optimal set of parameter estimates. Hence, the PARMS statement in the PROC NLIN procedure is written to assure a close starting value for D_0 in the iterative phase.

3. Comparison Between the Linear and Non-Linear Models

A question addressed was the propriety of using the PROC REG procedure under the assumption that the two-parameter Carleton damage function (D_0=1) was a reasonable model, and that the function, as previously discussed in Section V, can be linearized in order to estimate R_x and R_y. To answer this question, the procedure was used on a full size (1,600-observations) data set as provided by one of the P_k matrices. A printout and plot of the residuals obtained showed that the residuals ranged between ±0.2 and were particularly large (in absolute value) in the area of weapon impact (origin).

Using the same data set, PROC NLIN procedure was used under the assumption that the three-parameter Carleton damage function ($0 < D_0 < 1$) was a better model. In this instance, there was no need to manipulate the model as in the previous case or to discard values of P_k=0. The residuals obtained ranged between ±0.15 in the area of the origin and were much better elsewhere.

The poor fit in the neighborhood of the origin is due to the preponderance of the data points being outside that neighborhood. The points in the neighborhood of the origin are simply outnumbered by the other points. The values of P_k near the origin have almost no influence on the values of the parameters of the model.

For comparison purpose, a linearized regression analysis was performed on the data set corresponding to the following:

Weapon: 5EAKL, Target 8T1.5. Impact angle 15° Velocity 900 ft/sec. Using the three-parameter Carleton damage function. The residual sum of squares (RSS) obtained for the P_k values was 14.209. This was considerably more than the RSS obtained using the non-linear regression model, which was 3.269. The estimates of D_0, R_x, and R_y for the two models were as follows:

	3-parameter linear model	3-parameter non-linear model
D_0	0.020221	0.75515
R_x	99.570	72.9501
R_y	104.332	269.2381
RSS	14.209	3.269

In the least-square sense, the non-linear regression model yields much better parameter estimates than the linearized regression models. Also, as previously stated, the two-parameter linearized model yielding estimates for R_x and R_y obtained from PROC REG procedure were used as the initial values for PROC NLIN procedure.

SECTION VIII

COMPUTATIONAL ROUTINE

1. Introduction

In this section, the computational routine associated with SAS is detailed.

The primary input that is provided by the Air Force consists of the 40 x 40 probability of kill (P_k) matrix for a given weapon/target combination specified for a given weapon impact angle and impact velocity. This matrix is given as a computer printout.

2. Design Information

A copy of the computer program used appears in Appendix B.

In general, this program was designed to output all the desired estimates using both the linear and non-linear regression procedures, with the user only required to provide the input data. In addition, this program is written so that the input data set needs only contain 3 simple lines of code and the 1600 probability of kill (P_k) values typed (or punched) ten to a line (or card).

The majority of the code in this program is necessary only to:

(1) Manipulate, generate, label, or properly format the input and program generated data sets for the linear and non-linear analyses.

(2) Generate the properly formatted output.

Although a thorough understanding of SAS is necessary to comprehend these portions of the code, this should pose no problem in adapting this program for execution on other computers or at other facilities.

It should be noted, however, that the two INFILE statements in the program are used to read the input data set from a disk file and to read a

program - generated external disk file. Statements must be included in the JCL (for a batch run) or on the terminal (prior to an interactive run) to associate the correct disk files with the INFILE statements.

3. Input Data

The input data set may be entered onto a computer disk file or onto punchcards in the following format, each line representing one punchcard or one line on the terminal:

```
X  19.8     13.
Y  13.7     0.0
Z   0.0     0.0

. . . . . .     .0011     .0013     .0017 . . . . . .  .0001
                .0002     .0017     .0020 . . . . . .  .0007



160 lines



. . . . . .     .0001     .0007     .0008 . . . . . .  .0004
                <----------   10 values   ---------->
```

To input each separate data set, refer to the example above and do the following:

Line #1

The first entry is always the letter X. The second entry will be the difference between successive range values in the data set. Note that this value is a constant for each data set. The third entry will be computed as follows:

$$\left| \frac{\text{most negative range value}}{\text{difference between successive range values}} \right| + 1$$

which is always an integer.

Line #2

The first entry is the letter Y. The second entry is the difference between successive deflection values in the dataset. Note that this value is a constant for each data set. The third entry will be the number Ø.Ø.

Line #3

The first entry is the letter Z. The second and third entries are both the number Ø.Ø

Line #4 - Line #163

Refer to the P_k values as they are printed on the Air Force computer output. There are four pages, each with 400 P_k values. Each page contains the P_k values corresponding to 10 different deflection values and all 40 range values. Each of the 160 lines corresponds to one line of input data, going in order from first page to last, top to bottom.

4. Program Execution

The first part of the program consists of the code necessary to read in the input data set, generate the range and deflection values corresponding to each P_k value, and transpose all these values into the proper format for the analyses.

The second part of the program starts with PROC REG statement and MODEL statement necessary to call the PROC REG procedure and specify the model to be fitted, respectively. Following these are several more statements used for manipulation of the PROC REG output.

The next section begins with the statement;

PROC PRINTTO UNIT = 20;

(This causes the PROC NLIN non-linear regression output to be routed to an external disk file which is referenced later in the INFILE FT20F001 statement.) Following is a brief description of the statements used to perform the non-linear regression:

(1) PROC NLIN statement followed by the DATA = statement. The DATA = statement specifics the data set to be analyzed. A METHOD = statement may also be included specifying the mathematical procedure to be used. The default method is Gauss-Newton.

(2) PARMS statement followed by the RX = , RY = , and DO = assignments specifying the grid of initial parameter values to be used. The procedure computes the sums of squares for error for each possible combination and uses the parameter values in the grid corresponding to the least such sums of squares as the initial parameter values in the iterative procedure.

(3) BOUNDS statement followed by code designating the regions of consideration for each parameter estimate. The procedure will not consider parameter values outside the boundary regions during the iterative process.

(4) MODEL statement designating the model to be fitted, followed by statements of the form DER.DO = which give the partial derivatives of the model expression with respect to the parameters to be estimated.

Section 4, beginning with the statement

DATA TWO;

consists entirely of the code necessary to read the external file FT20F001 and manipulate the values there as necessary to compute the desired non-linear regression covariance estimates. SAS only automatically outputs the

required to find the values of the (asymptotic) variance and covariance estimates.

Finally, the last section, beginning with the statement

```
PROC PRINT DATA = FINALOUT LABEL;
```

is necessary to print out the data sets in which all the estimates have been stored.

5. Output Results

(a) For the linear regression procedure, the output results consist of the following:

- Estimates: R_x = ; R_y =
- Variance-covariance of R_x and R_y: $Var[R_x]$, $Var[R_y]$, $Cov[R_x\ R_y]$.
- Variance of $\ln(P_k)$ = (mean squared error)

(b) For the non-linear regression procedure the output results consist of the following:

- Estimates: D_0 = ; R_x = ; R_y =
- Variance - covariance estimates of D_0, R_x and R_y:

$Var[D_0]$, $Var[R_x]$, $Var[R_y]$, $Cov[R_x,R_y]$, $Cov[R_x,D_0]$, $Cov[R_y,D_0]$

-Variance of P_k = (mean squared error)

6. System Specifications

Hardware: IBM 4341 running VM/SP release 3 DASD 3370 (Disk Drive)

Operating System: VM/SP CMS

Programming Language: SAS79.6 with SAS ETS and SAS GRAPH

SECTION IX

COMPUTATIONAL RESULTS

In this section, using a selected set of 13 P_k matrices for four weapon/target situations at different impact angles ranging from 15° to 75°, at estimates of D_0, R_x, and R_y are given using the non-linear regression technique. The asymptotic variance-covariance matrix of D_0, R_x, and R_y is also given.

For reference purpose, the result of the linearized regression on the two-parameter Carleton damage function (D_0=1) is also given. This includes the estimates of R_x and R_y and the variance-covariance matrix of R_x and R_y.

1. Estimates of Weapon Parameters I

Weapon: 5EAKL   Target: 8T1.5   Impact Angle: 15°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 233.733 ft; R_y = 335.446 ft
- Variance-covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 16.4307 & \\ -4.8609 & 31.0638 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{6.29616} = 2.5092$

Non-Linear Regression

- Estimates: D_0 = .7551462; R_x = 72.9501 ft; R_y = 269.2381 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .000202 & & \\ .000250 & 1.3899 & \\ -.000300 & -1.7099 & 17.7441 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00204681} = .04524$

2. Estimates of Weapon Parameters 2

Weapon : 5EAKL   Target: 8T1.5   Impact Angle: 30°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 214.054 ft; R_y = 343.765 ft
- Variance-covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 14.1192 & \\ -5.3535 & 26.5136 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{5.8167} = 2.4118$

Non-Linear Regression

- Estimates: D_0 = .80309; R_x = 69.3650 ft; R_y = 292.5472 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .000227 & & \\ .000230 & 1.24858 & \\ -.000330 & -1.7550 & 20.8858 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00280587} = 0.05297$

3. Estimates of Weapon Parameters 3

Weapon : 5EAKL   Target: 8T1.5   Impact Angle: 45°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 184.038 ft; R_y = 381.716 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 9.1489 & \\ -9.4982 & 39.1425 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{4.7591} = 2.1815$

Non-Linear Regression

- Estimates: D_0 = .85879; R_x = 70.8659 ft; R_y = 318.4896 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00025 & & \\ .00023 & 1.2595 & \\ -.00035 & -1.8868 & 23.999 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00371501} = .06095$

4. Estimates of Weapon Parameters 4

Weapon : 5EAKL Target: 8T1.5 Impact Angle: 60°

Vel: 900 ft/sec

Linear Regression: ($D_0=1$)

- Estimates: R_x = 253.001 ft; R_y = 370.965 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 9.53298 & \\ -6.1764 & 23.1297 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{3.3698} = 1.8357$

Non-Linear Regression:

- Estimates: D_0 = .85805 ; R_x = 97.4700 ft; R_y = 317.1730 ft
- Variance - Covarianve matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00027 & & \\ .00035 & 2.5862 & \\ -.00038 & -2.8052 & 25.8300 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00418549} = .0647$

5. Estimates of Weapon Parameters 5

Weapon : 5EAKL Target: 8T1.5 Impact Angle: 75°

Vel: 900 ft/sec

Linear Regression: $(D_0=1)$

- Estimates: R_x = 328.089 ft; R_y = 362.266 ft;
- Variance - covariance matrix R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 12.5726 & \\ -4.0967 & 13.5213 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{.0105} = .10247$

Non-Linear Regression

- Estimates: D_0 = .92205; R_x = 143.0987 ft; R_y = 366.1414 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .000306 & & \\ .000510 & 5.45596 & \\ -.00040 & -4.3990 & 30.21041 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00743} = .086197$

6. Estimates of Weapon Parameters 6

Weapon : 5EAKL Target: 16 Impact Angle: 15°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 74.5391 ft; R_y = 112.712 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 3.61066 & \\ -.61024 & 4.28234 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{13.5915} = 3.6867$

Non-Linear Regression

- Estimates: D_0 = .33256; R_x = 12.9288 ft; R_y = 37.3862 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00092 & & \\ .0010 & .0100 & \\ -.0001 & -.0965 & .0727 \end{bmatrix}$$

- Root mean square see error on P_k: $\sqrt{.00029504} = .0172$

7. Estimates of Weapon Parameters 7

Weapon : 5EAKL   Target: 16   Impact Angle: 30°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 56.4879 ft; R_y = 118.59 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 1.2879 & \\ -.57349 & 3.55144 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{10.7703} = 3.2818$

Non-Linear Regression

- Estimates: D_0 = .367[illegible]34; R_x = 11.4333 ft; R_y = 41.9625 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00285 & & \\ .00239 & 1.99746 & \\ -.00293 & -.24438 & 23.5749 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.0105071} = .10250$

8. Estimates of Weapon Parameters 8

Weapon : 5EAKL   Target: 16   Impact Angle: 45°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 54.7351 ft; R_y = 123.717 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} .947699 & \\ -.69615 & 3.14925 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{8.62187} = 2.9363$

Non-Linear Regression

- Estimates: D_0 = .40749; R_x = 12.1322 ft; R_y = 44.6990 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00017 & & \\ .00014 & .11126 & \\ -.00017 & -.13663 & 1.3250 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00072634} = .02695$

9. Estimates of Weapon Parameters 9

Weapon : 5EAKL Target: 16 Impact Angle: 60°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 71.0691 ft; R_y = 124.729 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} .964991 & \\ -.58275 & 2.93055 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{8.23459}$ = 2.8696

Non- Linear Regression

- Estimates: D_0 = .44089; R_x = 17.1892 ft; R_y = 44.4199 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00018 & & \\ .00019 & .1944 & \\ -.00017 & -.1675 & 1.1318 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00075406}$ = .0275

10. Estimates of Weapon Parameters 10

Weapon : 5EAKL Target: 310 Impact Angle: 15°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 195.359 ft; R_y = 293.402 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 11.0944 & \\ -3.4227 & 22.8314 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{4.25801} = 2.0688$

Non-Linear Regression

- Estimates: D_0 = .82377; R_x = 78.1014 ft; R_y = 296.6297 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .000244 & & \\ .00022 & 1.6230 & \\ -.00029 & -2.0473 & 22.2833 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00420403} = .0648$

11. Estimates of Weapon Parameters 11

Weapon : 5EAKL   Target: 310   Impact Angle: 60°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 215.154 ft; R_y = 326.333 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 6.30579 & \\ -4.2457 & 17.9445 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{2.36543} = 1.5380$

Non-Linear Regression

- Estimates: D_0 = .90123; R_x = 121.5704 ft; R_y = 324.7570 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00030 & & \\ .00038 & 4.0974 & \\ -.00036 & -3.6502 & 27.9007 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00895887} = .09465$

12. Estimates of Weapon Parameters 12

Weapon : 5EAKL   Target: 310   Impact Angle: 75°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 285.001 ft; R_y = 321.833 ft
- Variance - covariance matrix of R_x and R_y:

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 9.25963 & \\ -2.8505 & 9.96642 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{1.97828} = 1.40651$

Non-Linear Regression

- Estimates: D_0 = .94998; R_x = 163.7629 ft; R_y = 359.1702 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00035 & & \\ .00039 & 7.82936 & \\ -.00041 & -5.7288 & 35.9268 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.01576} = .12554$

13. Estimates of Weapon Parameters 13

Weapon : 5EAKL   Target: 3172   Impact Angle: 75°

Vel: 900 ft/sec

Linear Regression: (D_0=1)

- Estimates: R_x = 220.38 ft; R_y = 237.948 ft
- Variance - covariance matrix of R_x and R_y

$$\begin{array}{c} R_x \\ R_y \end{array} \begin{bmatrix} 6.80197 & \\ -1.8223 & 5.66516 \end{bmatrix}$$

- Root mean square error on ln P_k: $\sqrt{6.61324} = 2.5716$

Non-Linear Regression

- Estimates: D_0 = .59485; R_x = 59.2070 ft; R_y = 122.9209 ft
- Variance - covariance matrix of D_0, R_x, and R_y:

$$\begin{array}{c} D_0 \\ R_x \\ R_y \end{array} \begin{bmatrix} .00029 & & \\ .00061 & 2.1304 & \\ -.00043 & -1.4743 & 8.2682 \end{bmatrix}$$

- Root mean square error on P_k: $\sqrt{.00238537} = .0488$

14. Summary of Estimates

	Weapon	Target	Impact Angle	D_0	R_x(ft)	R_y(ft)	σ_{D_0}	σ_{R_x} (ft)	σ_{R_y} (ft)	Roger's Ellipse Method D_0	R_x(ft)	R_y(ft)
1	SEAKL	8T1.5	15°	.75515	72.9501	269.2381	.01421	1.1789	4.2124	.79	75.1046	288.0415
2	SEAKL	8T1.5	30°	.80309	69.3650	292.5472	.01506	1.1174	4.5702	.84	75.3404	302.543
3	SEAKL	8T1.5	45°	.85879	70.8689	318.4896	.01582	1.1223	4.8989	.87	77.5703	329.5933
4	SEAKL	8T1.5	60°	.85805	97.4700	317.1730	.01647	1.6082	5.0823	.89	106.0595	330.1061
5	SEAKL	8T1.5	75°	.92205	143.0987	346.1414	.01750	2.3358	5.4964	.92	151.6255	361.922
6	SEAKL	16	15°	.33256	12.9288	37.3862	.0096	.3164	.8525	.42	11.0632	45.8650
7	SEAKL	16	30°	.36753	11.4333	41.9625	.05343	1.4133	4.8554	.47	10.0881	50.2876
8	SEAKL	16	45°	.40748	12.1322	44.6990	.01317	.3335	1.1511	.53	10.9063	5[illegible].4781
9	SEAKL	16	60°	.44089	17.1892	44.4199	.01331	.4410	1.0639	.52	16.1483	53.1698
10	SEAKL	310	15°	.82377	78.1014	296.6300	.01562	1.2740	4.720	.83	81.0504	304.0134
11	SEAKL	310	60°	.90123	121.5704	342.7570	.01732	2.0242	5.2821	.97	127.2942	326.6899
12	SEAKL	310	75°	.94998	163.7629	359.1702	.0188	2.7981	5.9939	1.01	163.9656	372.7855
13	SEAKL	3172	75°	.59485	59.2070	122.6209	.01703	1.4596	2.8754	.65	55.8542	142.6302

Example

Consider the following situation:

Weapon 5EAKL, Target 8T1.5, Impact angle 15°. The results of the non-linear regression analysis are:

$$\bar{D}_0 = .75515; \quad \bar{R}_x = 72.9501 \text{ ft}; \quad \bar{R}_y = 269.2381 \text{ ft}$$

$$\sigma_{D_0} = .01414; \quad \sigma_{R_x} = 1.1784 \text{ ft}; \quad \sigma_{R_y} = 4.2124 \text{ ft}$$

The two-standard deviation confidence interval on the weapon parameters will be

$$\bar{D}_0 \pm 2\,\sigma_{D_0} = .75515 \pm .02828$$

$$\bar{R}_x \pm 2\,\sigma_{R_x} = 72.9501 \pm 2.3568 \text{ ft}$$

$$\bar{R}_y \pm 2\,\sigma_{R_y} = 269.2381 \pm 8.4248 \text{ ft}$$

SECTION X

COMPUTATION OF ERROR IN P_k

Consider again the three-parameter Carleton damage function.

$$P_k = P_k(D_0, R_x, R_y)$$

$$= D_0 \exp\left[-D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \tag{53}$$

Given estimates of D_0, R_x, and R_y, say $\bar{D}_0$, $\bar{R}_x$, and $\bar{R}_y$, as well as their variance-covariance matrix, it is required to find an estimate of $E[P_k]$ and $Var[P_k]$ (which is a measure of the error on the estimate of P_k), at a particular target location (x,y).

The methodology outlined in [6] will be applied here using the Taylor's series estimation method.

1. Estimation of $E[P_k]$

As a first approximation one can write

$$E[P_k] = P_k(\bar{D}_0, \bar{R}_x \bar{R}_y)$$

$$= \bar{D}_0 \exp\left[-\bar{D}_0\left(\frac{x^2}{\bar{R}_x^2} + \frac{y^2}{\bar{R}_y^2}\right)\right] \tag{54}$$

2. Estimation of Var$[P_k]$

Expanding $P_k = P_k(D_0, R_x, R_y)$ as a Taylor's series about the point $(\bar{D}_0, \bar{R}_x, \bar{R}_y)$ and retaining only the first order terms result in:

$$P_k(D_0, R_x, R_y) = P_k(\bar{D}_0, \bar{R}_x, \bar{R}_y)$$

$$+ (D_0 - \bar{D}_0)\left(\frac{\partial P_k}{\partial D_0}\right)_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ (R_x - \bar{R}_x)\left(\frac{\partial P_k}{\partial R_x}\right)_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ (R_y - \bar{R}_y)\left(\frac{\partial P_k}{\partial R_y}\right)_{\bar{D}_0, \bar{R}_x, \bar{R}_y} \qquad (55)$$

Transposing and squaring both sides of (55) yields

$$[P_k(D_0, R_x R_y) - P_k(\bar{D}_0, \bar{R}_x, \bar{R}_y)]^2$$

$$= (D_0 - \bar{D}_0)^2 \left(\frac{\partial P_k}{\partial D_0}\right)^2_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ (R_x - \bar{R}_x)^2 \left(\frac{\partial P_k}{\partial R_x}\right)^2_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ (R_y - \bar{R}_y)^2 \left(\frac{\partial P_k}{\partial R_x}\right)^2_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ 2(D_0 - \bar{D}_0)(R_x - \bar{R}_x)\left[\left(\frac{\partial P_k}{\partial D_0}\right)\left(\frac{\partial P_k}{\partial R_x}\right)\right]_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ 2(D_0 - \bar{D}_0)(R_y - \bar{R}_y)\left[\left(\frac{\partial P_k}{\partial D_0}\right)\left(\frac{\partial P_k}{\partial R_y}\right)\right]_{\bar{D}_0, \bar{R}_x, \bar{R}_y}$$

$$+ 2(R_x - \bar{R}_x)(R_y - \bar{R}_y)\left[\left(\frac{\partial P_k}{\partial R_x}\right)\left(\frac{\partial P_k}{\partial R_y}\right)\right]_{\bar{D}_0, \bar{R}_x, \bar{R}_y} \qquad (56)$$

Taking expectations on both sides of (56), one obtains as a first approximation:

$$\begin{aligned} \mathrm{Var}[P_k] \approx\ & \mathrm{Var}[D_0] \left(\frac{\partial P_k}{\partial D_0}\right)^2_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y} \\ & + \mathrm{Var}[R_x] \left(\frac{\partial P_k}{\partial R_x}\right)^2_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y} \\ & + \mathrm{Var}[R_y] \left(\frac{\partial P_k}{\partial R_y}\right)^2_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_x} \\ & + 2\ \mathrm{Cov}[D_0,R_x] \left[\left(\frac{\partial P_k}{\partial D_0}\right)\left(\frac{\partial P_k}{\partial R_x}\right)\right]_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y} \\ & + 2\ \mathrm{Cov}[D_0,R_y] \left[\left(\frac{\partial P_k}{\partial D_0}\right)\left(\frac{\partial P_k}{\partial R_y}\right)\right]_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y} \\ & + 2\ \mathrm{Cov}[R_x,R_y] \left[\left(\frac{\partial P_k}{\partial R_x}\right)\left(\frac{\partial P_k}{\partial R_y}\right)\right]_{\bar{D}_0,\ \bar{R}_x,\ \bar{R}_y} \end{aligned} \tag{57}$$

The expressions for the partial derivatives are given in (46), (47), and (48) and are repeated here:

$$\frac{\partial P_k}{\partial D_0} = \left[1 - D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \quad \exp\left[- D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \tag{58}$$

$$\frac{\partial P_k}{\partial R_x} = 2x^2 \frac{D_0^2}{R_x^3} \quad \exp\left[-D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \tag{59}$$

$$\frac{\partial P_k}{\partial R_y} = 2y^2 \frac{D_0^2}{R_y^3} \quad \exp\left[-D_0\left(\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2}\right)\right] \tag{60}$$

set

$$\lambda = D_0\left(\frac{x^2}{\bar{R}_x^2} + \frac{y^2}{\bar{R}_y^2}\right) \tag{61}$$

Substituting (58), (59), (60), and (61) in (57) yields

$$
\begin{aligned}
\mathrm{Var}[P_k] = {} & (1-\lambda)^2 e^{-2\lambda} \mathrm{Var}[D_0] + 4x^4 \frac{\bar{D}_0^4}{\bar{R}_x^6} e^{-2\lambda} \mathrm{Var}[R_x] \\
& + 4y^4 \frac{\bar{D}_0^2}{\bar{R}_y^6} e^{-2\lambda} \mathrm{Var}[R_y] \\
& + 4x^2 \frac{\bar{D}_0^2}{\bar{R}_x^3} (1-\lambda) e^{-2y} \mathrm{Cov}[D_0, R_x] \\
& + 4y^2 \frac{\bar{D}_0^2}{\bar{R}_y^3} (1-\lambda) e^{-2\lambda} \mathrm{Cov}[D_0, R_y] \\
& + 8x^3 y^2 \frac{\bar{D}_0^4}{\bar{R}_x^3 \bar{R}_y^3} e^{-2\lambda} \mathrm{Cov}[R_x, R_y]
\end{aligned}
\tag{62}
$$

Example

Given: weapon 5EAKL, target 8T1.5, impact angle 15°, the following are sample data from the P_k matrix:

x \ y	97.2	121.5
37.9	.8974	.8764
75.8	.4263	.3920

x \ y	97.2	121.5
-37.9	.2470	.1770
-75.8	.0683	.0555

Estimates of $E[P_k]$ using Roger's Ellipse Program

x \ y	97.2	121.5
37.9	.5905	.5613
75.8	.3229	.3070

Estimates of $E[P_k]$ using Non-Linear Regression Procedure

x \ y	97.2	121.5
37.9	.5582	.5281
75.8	.3028	.2865

The estimated $E[P_k]$'s are the same for both positive and negative values of the x's.

Example

For the following: weapon 5EAKL, target 8T1.5, impact angle 15°, the P_k matrix provides the following points:

x \ y	97.2	121.5
37.9	.8974	.8764
75.8	.4263	.3920

To find P_k at x=50 ft and y=100 ft, an interpolation scheme is used. At x=37.9 ft, interpolation yields the following P_k value at y=100 ft.

$$P_k = .8974 + \frac{(100 - 97.2)\ (.8964 - .8974)}{(121.5 - 97.2)}$$

$$= .8974 + \frac{(2.8)(-.0210)}{(24.3)} = .8950$$

At x=75.8 ft, interpolation yields the following P_k value at y=100 ft.

$$P_k = .4263 + \frac{(100 - 97.2)(.3920 - .4263)}{(121.5 - 97.2)}$$

$$= .4263 + \frac{(2.8)(-.0343)}{(24.3)} = .4223$$

Interpolation at y=100 ft yields for x=50 ft.

$$P_k = .8950 + \frac{(50 - 37.9)\ (.4223 - .8950)}{(75.8 - 37.9)}$$

$$= .8950 + \frac{(121.1)\ (-.4727)}{(37.9)} = .7441$$

Example

For the following: weapon 5EAKL, target 8T1.5, impact angle 15°, the P_k matrix provides the following data points:

x \ y	97.2	121.5
-37.9	.2470	.1770
-75.8	.0683	.0555

The absolute values of x will be used in the sequel. To find P_k at x=50 ft and y=100 ft, an interpolation scheme is used. At x=37.9 ft, interpolation yields at the following P_k value at y=100 ft.

$$P_k = .2470 + \frac{(100 - 97.2)\ (.1770 - .2470)}{(121.5 - 97.2)}$$

$$= .2470 + \frac{(2.8)\ (-.0700)}{(24.3)} = .2389$$

At x=75.8 ft, interpolation yields the following P_k value at y=100 ft

$$P_k = .0683 + \frac{(100 - 97.2)\ (.0555 - .0683)}{(121.5 - 97.2)}$$

$$= .0683 + \frac{(2.8)\ (-.0128)}{(24.3)} = .0668$$

Interpolation at y=100 ft yields for x = 50 ft:

$$P_k = .2389 + \frac{(50 - 37.9)\ (.0668 - .2389)}{(75.8 - 37.9)}$$

$$= .2389 + \frac{(12.1)\ (-.1721)}{(37.9)} = .1840$$

Note the extreme differences in P_k between (50,100) and (-50,100).

Example

For the following: weapon 5EAKL, target 8T1.5, impact angle 15°, Roger's ellipse program yields

$$D_0 = .79;\ R_x = 75.1046 \text{ ft};\ R_y = 288.0415 \text{ ft}.$$

The $E[P_k]$ value for x=50 ft and y=100 ft is

$$E[P_k] = (.79) \exp [(-.79\ [(\frac{50}{75.1046})^2 + (\frac{100}{288.0415})^2]]$$

$$= (.79\ (.640599) = .50607$$

Example

For the following: weapon 5EAKL, target 8T1.5, impact angle 15°, the two-parameter linearized Carleton damage function (D_0=1) model yields:

$$R_x = 233.733 \text{ ft, and } R_y = 335.446 \text{ ft}.$$

The $E[P_k]$ value for x =50 ft and y=100 ft is

$$E[P_k] = \exp [-\ (\frac{x^2}{R_x^2} + \frac{y^2}{R_y^2})]$$

$$= \exp \{-\ [(\frac{50}{233.733})^2 + (\frac{100}{335.446})^2]]$$

$$= \exp(-\ .13463) = .87404$$

3. Example

Consider the results given in Section IX.1 for the following: weapon 5EAKL, target 8T1.5, impact angle 15°.

Here

$\bar{D}_0$ = .7551462; $\bar{R}_x$ = 72.9501 ft; $\bar{R}_y$ = 269.2381 ft

Var[D_0] = .00020; Var[R_x] = 1.3899 ft^2; Var[R_y] = 17.7441 ft^2

Cov[D_0,R_x] = .00025; Cov[D_0,R_y] = -.00030; Cov[R_x,R_y] = -1.7099

It is required to determine an estimate of E[P_k] with a two standard deviation confidence interval for a point target located at x=50 ft and y=100 ft. The non-linear regression model is used here.

First λ is computed

$$\lambda = .7551462\left[\left(\frac{50}{72.9501}\right)^2 + \left(\frac{100}{269.2381}\right)^2\right]$$

$$= .7551462\ (.4697731874 + .1379516711)$$

$$= .458921$$

From (54) one obtains

$$E[P_k] = \bar{D}_0\ e^{-\lambda}$$

$$= (.7551462)\ e^{-.458921}$$

$$= .4772$$

From (62) $Var[P_k]$ is computed

$$Var[P_k] = (1 - .458921)^2 e^{-2(.458921)} (.00020)$$

$$+ 4(50)^4 \frac{(.7551462)^4}{(72.9501)^6} e^{-2(.458921)} (1.3899)$$

$$+ 4(100)^4 \frac{(.7551462)^4}{(269.2381)^6} e^{-2(.458921)} (17.7441)$$

$$+ 4(50)^2 \frac{(.7551462)^2}{(72.9501)^3} (1 - .458921) e^{-2(.458921)} (.00025)$$

$$+ 4(100)^2 \frac{(.7551462)^2}{(269.2381)^3} (1 - .458921) e^{-2(.458921)} (-.00030)$$

$$+ 8(50)^2(100)^2 \frac{(.7551462)^4}{(72.9501)^3(2692381)^3} e^{-2(.458921)} (-1.7099)$$

$$Var[P_k] = .000023385 + .000029942 + .000002420 + .000000794$$

$$- .000000140 - .000005862$$

$$= .000,050,539$$

The standard deviation on the estimated P_k is

$$\sigma_{P_k} = \sqrt{.000,050,539} = .0071$$

A two-standard deviation confidence interval on $\hat{P}_k$, the true mean value of P_k is

$$\hat{P}_k = E[P_k] \pm 2\sigma_{P_k} = .4772 \pm .0142$$

This means that at least 75 percent of the future values of $\hat{P}_k$, at x=50 ft, y=100 ft, should lie in the computed interval.

SECTION XI

CONCLUSIONS AND RECOMMENDATIONS

There are two main conclusions that can be reached at this stage:

1. Roger's ellipse program provides a good approximation to the estimate of the three parameter Carleton damage function. These estimates compare favorably with the estimates obtained from a non-linear regression procedure.

2. The results indicate that in modeling the probability of kill for low impact angles, it might be useful to consider either modifications of the Carleton damage function or other functions. One particular suggestion is to fit two Carleton damage functions, one for positive ranges and one for negative ranges.

Some of the problems suggested to be studied in future work are:

1. The use of a non-linear regression analysis to fit a blast damage function to actual data.

2. The estimation of the errors in P_k when fragmentation and blast are simultaneously present.

3. The development of a methodology to identify the sources of errors and to measure the magnitudes of these errors which are inherently present in the elements of the P_k matrix. This would require a thorough understanding and analysis of the methodology used in

   a. the fragmentation testing program for weapons;

   b. the target vulnerability program.

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APPENDIX A
ROGER'S ELLIPSE PROGRAM

```
C     ROGER'S ELLIPSE PROGRAM
      ASTH = 0.
      DO 210 I=1,36
      RBSUM1 = 0.
      R2SUM1 = 0.
      READ(1,END=355) (PK(J),J-1,NR)
      DO 200 J = 1,NR
      R1D = R1(J+1)-R1(J)
      R1S = R1(J+1)+R1(J)
      RBS = PK(J)*R1D
      RBSUM1 = RBSUM1 + RBS
      R2S = RBS * R1S
      R2SUM1 = R2SUM1 + R2S
  200 CONTINUE
      IF(R2SUM1.EQ.0.) S2TH = 1.
      IF(R2SUM1.EQ.0.) GO TO 201
      S2TH = 1.-RBSUM1**2/R2SUM1
  201 ASTH = ASTH + S2TH
      RTH = SQRT(R2SUM1)
      RT(I) = RTH
      STH = SQRT(S2TH)
  210 CONTINUE
      REWIND 1
      CONST = 4.0/3.14159265
      AA = ASTH/36.
      DO = CONST*(1.-AA)
      NZ = 36
      PI = 3.14159265
      CA = 0.
      Q = 0.
      SA = 1.
      DO 300 I = 1,NZ
      AI = I
      TH = 5.*(AI-.5)
      TH1 = TH/57.29578
      ST(I) = SIN(TH1)
      ST2(I) = ST(I)**2
      CT(I) = COS(TH1)
      CT2(I) = CT(I)**2
  300 CONTINUE
  320 D = 0.
      DO 400 I = 1,NZ
      TI = SQRT(CMAE/(PI*(SA*ST2(I)+CT2(I)/SA)))
      WI = TI**3*((SA*ST(I))**2-CT2(I))/SA**2
      RMT = RT(I) - TI
      D = D + (RMT*WI)
```

```
400 CONTINUE
    IF(D.GT.0.) GO TO 309
    CA = SA
    SA = SA + 1.
    GO TO 320
309 CONTINUE
310 S = 0.
    D = 0.
    DO 500 I = 1,NZ
    TI = SQRT(CMAE/(PI*(SA*ST2(I)+CT2(I)/SA)))
    WI = TI**3*((SA*ST(I))**2-CT2(I))/SA**2
    RMT = RT(I) - TI
    S = S + RMT**2
    D = D + (RMT*WI)
500 CONTINUE
    S = 2.*S
    IF(ABS(Q-SA).LE..001) GO TO 5000
    IF(D.GT.0.) GO TO 330
    CA = SA
    GO TO 340
330 C = SA
340 Q = SA
    SA = (CA + Q)*.5
    GO TO 310
5000 SA = 1./SA
    WRITE (6,9) SA
    RAD = SQRT(CMAE/PI)
    RSA = SQRT(SA)
    RX = RAD/RSA
    RY = RAD/RSA
    WRITE(6,350)  RAD
    WRITE(6,351)  RSA
    WRITE(6,352)  RX
    WRITE(6,353)  RY
360 FORMAT(1H0,8X,'DO = ',F10.2)
    WRITE(15) RX,RY,DO
    REWIND 1
```

APPENDIX B

SAS COMPUTER PROGRAM FOR ESTIMATING
PARAMETER VALUES OF P_K

```
*---------------------------------------------------------------------*;
                    *PART 1
  THIS SECTION CONSISTS OF:
  (1) A DATA STEP TO READ THE INPUT DATA, GENERATE THE RANGE AND
      DEFLECTION VALUES CORRESPONDING TO EACH 'PROBABILITY OF KILL'
      VALUE (Z), AND CREATE A DATA SET WITH THE RANGE, DEFLECTION,
      AND Z VALUES EACH STORED IN ONE OBSERVATION.  I.E. 3 TOTAL
      OBSERVATIONS, EACH CONSISTING OF AN IDENTIFICATION (ID)
      VALUE (X,Y,OR Z) AND 1600 DATA VALUES.
  (2) A PROC STEP TO TRANSPOSE THE ABOVE DATA SET INTO A DATA SET
      WITH 1600 OBSERVATIONS, EACH CONTAINING ONE Z VALUE AND THE
      CORRESPONDING X AND Y VALUES.
  (3) A DATA STEP TO ADD THE VALUES LOG(Z) (Z.GT.0), X**2, AND
      Y**2 TO EACH OBSERVATION.;
*---------------------------------------------------------------------*;
DATA TEMP,WPNDAT;
   ARRAY K K1-K1600;
   INFILE MYDATA;
   INPUT IS $ INT  JNUM ;
   IF ID='Y' THEN DO;
      DO I=1 TO 4;
      DO J=1 TO 40;
      DO L=1 TO 10;
      _I_=L+400*(I-1)+10*(J-1);
      K=(10*(I-1)+L)*INT;
      END; END; END;
      OUTPUT;
      END;
   IF ID='X' THEN DO;
      DO I=1 TO 4;
      DO J=1 TO 40;
      DO L=1 TO 10;
      _I_=L+400*(I-1)+10*(J-1);
      K=(J-JNUM)*INT;
      END; END; END;
      OUTPUT;
      END;
   IF ID='Z' THEN DO;
      INPUT K1-K1600;
      OUTPUT;
      END;
   RUN;
PROC TRANSPOSE OUT=TPOSE;
   VAR K1-K1600;
   ID ID;
   RUN;
DATA LINPOSE;
   SET TPOSE;
   XS=X*X;
   YS=Y*Y;
   IF Z GT 0 THEN LZ=-LOG(Z);
   ELSE LZ=.;
   RUN;
```

```
  *--------------------------------------------------------------*;
                *PART 2
     THIS SECTION CONSIST OF:
     (1) THE "PROC REG' PROCEDURE STATEMENT NECESSARY TO RUN THE
         DESIRE LINEAR REGRESSION ON THE FINAL DATA SET CREATED IN
         PART 1 ABOVE.  THE 'OUTEST=VCOVMAT' OPTION IN THIS STATEMENT
         CAUSES THE PROCEDURE TO OUTPUT A DATA SET CONTAINING THE
         REGRESSION PARAMETER ESTIMATES.
     (2) TWO DATA STEPS WITH STATEMENTS NECESSARY TO CREATE A DATA
         SET CONTAINING THE PROPERLY LABELED PARAMETER ESTIMATES.
     (3) A THIRD DATA STEP TO PERFORM THE COMPUTATIONS FOR THE
         LINEAR REGRESSION ESTIMATES OF RX, RY, VAR(RX), VAR(RY), AND
         COV(RX,RY).
     (4) A FOURTH DATA STEP TO CREATE A 1600 OBSERVATION DATA SET,
         EACH OBSERVATION CONTAINING ONE DATA POINT (X, Y, Z) AND
         THE RX AND RY ESTIMATES FOUND ABOVE.;
   *--------------------------------------------------------------*;
PROC REG OUTEST=VCOVMAT COVOUT;
   MODEL LZ=XS YS/NOINT NOPRINT;
   OUTPUT OUT=WPNINFO P=LZHAT;
   RUN;
DATA TX TY TP;
   SET VCOVMAT;
   IF _MODEL_='XS' THEN OUTPUT TX;
   IF _MODEL_='YS' THEN OUTPUT TY;
   IF _MODEL_='  ' THEN OUTPUT TP;
   RUN;
DATA OUT;
   MERGE TY (RENAME=(YS=VARBHAT))
         TX (RENAME=(XS=VARAHAT YS =COVAB))
         TP (RENAME=(XS=AHAT YS=BHAT));
   RUN;
DATA FINALOUT(KEEP=RXHAT RYHAT SIGMASQ VARRXHAT VARRYHAT COVRXRY);
   SET OUT;
   RXHAT=SQRT(1/AHAT);
   RYHAT=SQRT(1/BHAT);
   SIGMASQ=_SIGMA_**2;
   VARRXHAT=VARAHAT/(4*AHAT**3);
   COVRXRY=COVAB/(4*(AHAT*BHAT)**1.5);
   VARRYHAT=VARBHAT/(4*BHAT**3);
   CALL SYMPUT('JX',RXHAT);
   CALL SYMPUT('JY',RYHAT);
   RUN;
DATA JPOSE(KEEP=X Y Z RXHAT RYHAT);
   IF _N_=1 THEN SET FINALOUT;
   SET LINPOSE;
   RUN;
```

```
*---------------------------------------------------------------------*;
                   *PART 3
    THIS SECTION CONSISTS OF:
    THE STATEMENTS NECESSARY TO CALL THE PROC NLIN PROCEDURE,
    PRECEEDED AND FOLLOWED BY 'PROC PRINTTO' STATEMENTS. THE 'PROC
    PRINTTO' STATEMENTS CAUSE THE PROCEDURE OUTPUT FROM PROC NLIN TO
    BE WRITTEN TO AN EXTERNAL FILE WHICH IS REFERENCED LATER IN THE
    PROGRAM WITH AN 'INFILE' STATEMENT.  THE EXTERNAL FILE WILL HAVE
    THE NAME 'FT20F001'.
    NOTE:  THIS IS NECESSARY SINCE THE PROC NLIN PROCEEDURE OUTPUT
           INCLUDES CORRELATION BUT NOT COVERIANCE ESTIMATES.  THE
           EXTERNAL FILE IS READ BACK INTO A SAS DATA SET (PART 4)
           AND THE COVARIANCE ESTIMATES ARE COMPUTED FROM THE
           CORRELATION ESTIMATES AND PARAMETER STANDARD ERRORS WHICH
           ARE PICKED OUT AND LABELED.;
*---------------------------------------------------------------------*;

PROC PRINTTO UNIT=20;
   RUN;
PROC NLIN DATA=JPOSE;
   PARMS RX=&JX
         DO=0.5 TO 1.3 BY .05
         RY=&JY;
   BOUNDS
         0<RX<=900
         0<DO<=1.3
         0<RY<=900;
   TEMP= -DO*(X**2/RX**2 + Y**2/RY**2);
   MODEL Z=DO*EXP(TEMP);
   DER.RX= (DO**2)*EXP(TEMP)*2*(X**2)/(RX**3);
   DER.RY= (DO**2)*EXP(TEMP)*2*(Y**2)/(RY**3);
   DER.DO=EXP(TEMP)*(1+TEMP);
   RUN;
PROC PRINTTO;
   RUN;
```

```
*-----------------------------------------------------------------*;
                    *PART 4
     THIS SECTION CONSISTS OF:
     THE DATA STEPS AND PROC STEPS NECESSARY TO READ IN THE EXTERNAL
     FILE CREATED IN PART 3 ABOVE AND CREATE A FINAL DATA SET WHICH
     CONTAINS THE NON-LINEAR PARAMETER, VARIANCE. AND COVARIANCE
     ESTIMATES.;
*-----------------------------------------------------------------*;
DATA TWO;
   INFILE FT20F001;
   INPUT PID $ ;
   IF PID='RX' THEN DO;
      INPUT P1 ?? ;
      IF P1=. THEN DO;
      INPUT; DELETE;
   END;
   ELSE DO;
   INPUT P2 P3; OUTPUT;
   END;
   END;
   IF PID='RY' THEN DO;
      INPUT P1-P3; OUTPUT;
   END;
   IF PID='DO' THEN DO;
      INPUT P1-P3; OUTPUT;
   END;
   IF PID='RESIDUAL' THEN DO;
      INPUT P1-P3; OUTPUT;
   END;
   RUN;
PROC SORT DATA=TWO;
   BY PID;
   RUN;
DATA BA(KEEP=RX STDRX) BB(KEEP=RY STDRY) BC(KEEP=DO STDDO)
     BD(KEEP=MSE) BE(KEEP=CORXD CORXY) BF(KEEP=CORYD);
   SET TWO;
   BY PID;
   IF FIRST.PID THEN DO;
      IF PID='RY' THEN DO;
      RY=P1;
      STDRY=P2;
      OUTPUT BB;
      END;
      IF PID='RX' THEN DO;
      RX=P1;
      STDRX=P2;
      OUTPUT BA;
      END;
      IF PID='DO' THEN DO;
      DO=P1;
      STDDO=P2;
      OUTPUT BC;
      END;
      IF PID='RESIDUAL' THEN DO;
      MSE=P3;
      OUTPUT BD;
      END;
   END;
```

```
   IF LAST.PID THEN DO;
      IF PID='RX' THEN DO;
      CORXD=P2;
      CORXY=P3;
      OUTPUT BE;
      END;
      IF PID='RY' THEN DO;
      CORYD=P2;
      OUTPUT BF;
      END;
   END;
   RUN;
DATA ATWO;
   MERGE BA BB BC BD BE BF;
   RUN;
DATA AFINAL;
   SET ATWO;
   VARRX=STDRX*STDRX;
   VARRY=STDRY*STDRY;
   VARDO=STDDO*STDDO;
   COVRXRY=STDRX*STDRY*CORXY;
   COVRXDO=STDRX*STDDO*CORXD;
   COVRYDO=STDRY*STDDO*CORYD;
     KEEP VARRX VARRY VARDO COVRXDO COVRYDO COVRXRY MSE RX RY DO;
   RUN;
   *----------------------------------------------------------------------*;
                  *PART 5
      THIS SECTION CONSISTS OF:
      THE 'PROC PRINT' PROCEEDURES NECESSARY TO PRINT OUT THE LINEAR
      AND NON-LINEAR ESTIMATES IN THE PROPER FORMAT.;
   *----------------------------------------------------------------------*;
PROC PRINT DATA=FINALOUT LABEL;
   VAR RXHAT RYHAT SIGMASQ VARRXHAT VARRYHAT COVRXRY;
   TITLE1 LINEAR REGRESSION ESTIMATORS;
   TITLE2 MODEL... -LOG(P)=X**2*(1/RX)**2+Y**2*(1/RY)**2;
   LABEL RXHAT=RX RYHAT=RY SIGMASQ=ERROR VARIANCE
         VARRXHAT=VAR(RX) VARRYHAT=VAR(RY) COVRXRY=COV(RX,RY);
   RUN;
PROC PRINT DATA=AFINAL LABEL;
   VAR DO RX RY MSE VARRX VARRY VARDO
       COVRXRY COVRXDO COVRYDO;
   TITLE1 NON-LINEAR REGRESSION ESTIMATORS;
   TITLE2 MODEL...P=DO*EXP(-DO*(X**2/RX**2+Y**2/RY**2);
   LABEL VARRX=VAR(RX) VARRY=VAR(RY) VARDO=VAR(DO)
         MSE=ERROR_VARIANCE
         COVRXRY=COV(RX,RY) COVRXDO=COV(RX,DO) COVRYDO=COV(RY,DO);
   RUN;
```


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